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ON STRETCHING, BENDING, TWISTING AND FLEXURE OF CYLINDRICAL SHELLS†

E. REISSNER

Department of Applied Mechanics and Engineering Sciences, University of California San Diego, La Jolla, CA 92093, U.S.A.

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Abstract—The first objective of this work is to use the differential equations of equilibrium of cylindrical shells in conjunction with the principle of minimum complementary energy for the derivation of constitutive equations for coupled stretching, bending and twisting of straight thin-walled open- and closed-cross section beams.

The second objective is to use appropriate solutions of the cylindrical-shell differential equations, again in conjunction with the principle of minimum complementary energy, for the sake of deriving values for flexibility and stiffness coefficients of end-loaded, thin-walled cantilever beams, with applications to the problem of shear center and twist center location, including an analysis of the effect of shear lag on the location of these centers.

INTRODUCTION

In what follows we are concerned with simplifications and generalizations of results for pure bending, stretching, twisting and flexure of thin-walled beams, including the problem of shear center and twist center location, which have been obtained recently[3–6].

For the problems of bending, stretching and twisting our principal simplification consists in the use of an appropriate version of the principle of minimum complementary energy in place of the explicit consideration of strain displacement relations[3, 4, 6].

Insofar as the problem of cantilever flexure is concerned we are able to simplify the problem for an important class of cases by recognizing that application of the complementary energy principle in Rayleigh–Ritz fashion can be done in such a way that it leads directly to the appropriate explicit form of load deflection relations which are involved in our defining relations for shear center and twist center location[5].

Insofar as the generalizations of our earlier work are concerned we mention in particular a consideration of the flexure problem without use of a Euler-Bernoulli hypothesis concerning the distribution of axial normal strains, for the purpose of appraising the influence of an effect known as *shear lag* on the location of the centers of shear and of twist.

DIFFERENTIAL EQUATIONS OF EQUILIBRIUM FOR CYLINDRICAL SHELLS AND EXPRESSIONS FOR CROSS-SECTIONAL FORCES AND MOMENTS

We consider cylindrical shells with generators in the direction of a z-axis, and with the surface equation of the shell given in the form $x_1 = x_1(s)$, $x_2 = x_2(s)$, where s represents arc length in circumferential direction.

We designate stress resultants by N and Q, and stress couples by M, in accordance with Fig. 1, and have then three equations of force equilibrium and three equations of moment equilibrium of the form

$$N_{ss,s} + N_{zs,z} + kQ_s = 0, \qquad N_{sz,s} + N_{zz,z} = 0, \tag{1, 2}$$

$$Q_{s,s} + Q_{z,z} - kN_{ss} = 0, \qquad N_{sz} - N_{zs} + kM_{sz} = 0, \tag{3, 4}$$

$$M_{ss,s} + M_{zs,z} = Q_s, \qquad M_{sz,s} + M_{zz,z} = Q_z. \tag{5, 6}$$

In this $k = \mathrm{d}\theta/\mathrm{d}s$ represents the circumferential curvature of the shell.

Expressions for cross-sectional forces and moments are given, again in accordance with Fig. 1, as integrals over stress resultants and couples, as follows

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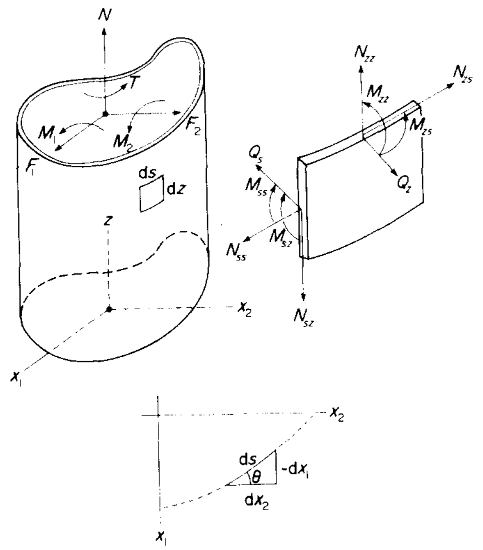

Fig. 1.

$$N=\int N_{zz}\,\mathrm{d}s,\qquad M_1=\int (N_{zz}x_2-M_{zz}x_1')\,\mathrm{d}s, \tag{7, 8}$$

$$M_2=-\int (N_{zz}x_1+M_{zz}x_2')\,\mathrm{d}s,\qquad F_1=\int (N_{zs}x_1'+Q_zx_2')\,\mathrm{d}s, \tag{9, 10}$$

$$F_2=\int (N_{zs}x_2'-Q_zx_1')\,\mathrm{d}s,\qquad T=\int (M_{zs}+r_NN_{zs}-r_QQ_z)\,\mathrm{d}s, \tag{11, 12}$$

where

$$(\ )'\equiv \mathrm{d}(\ )/\mathrm{d}s,\qquad x_1'=-\sin\theta,\qquad x_2'=\cos\theta, \tag{13}$$

and

$$r_N=x_1x_2'-x_2x_1',\qquad r_Q=x_1x_1'+x_2x_2', \tag{14}$$

with the differentiation formulas, $r_N'=kr_Q$ and $r_Q'=1-kr_N$.

The integrals in eqns (7)–(12) are to be considered as line integrals over a closed contour, for the case of closed-cross section shells. The case of open-cross section shells may be considered to be a special case of this, with resultants and couples having zero values along appropriate portions of the contour.

It is an important property of the integrals in eqns (10)–(12) that their direct usefulness depends on the assumption of a non-vanishing transverse shear deformability of the material of the shell. Upon introduction of the Love-Kirchhoff hypothesis of no transverse shear deformation into theory, with consequential reactive force properties for the resultants Q_z and Q_s, it becomes advantageous to eliminate Q_z from eqns (10)–(12). Use of the equilibrium eqn (6), in conjunction with an integration by parts and observation of eqn (4), then gives as expressions for F_1, F_2 and T,

$$F_1=\int (N_{sz}x_1'+M_{zz,z}x_1)\,\mathrm{d}s,\qquad F_2\int (N_{sz}x_2'+M_{zz,z}x_2)\,\mathrm{d}s, \tag{15, 16}$$

$$T=\int (M_{sz}+M_{zs}+r_NN_{sz}-R_QM_{zz,z})\,\mathrm{d}s, \tag{17}$$

and we note, in particular, the replacement of the shear-stress resultant N_{zs} in (9)–(12) by the shear-stress resultant N_{sz} in (15)–(17).

THE PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY FOR A CYLINDRICAL SHELL

We supplement the equilibrium differential eqns (1)–(6) by constitutive equations, in terms of a complementary energy density function W, of the form

$$\epsilon_{zz} = \frac{\partial W}{\partial N_{zz}},\quad \epsilon_{zs} = \frac{\partial W}{\partial N_{zs}},\dots,\gamma_z = \frac{\partial W}{\partial Q_z},\dots,\kappa_{ss} = \frac{\partial W}{\partial M_{ss}}, \tag{18}$$

and consider the case of a closed cross section shell with two edges $z = z_1$ and $z = z_2$ of the shell having prescribed values of the displacements u_z, u_s, w, φ_z, φ_s, with the meaning of these components being such as to be consistent with an expression $N_{zz}u_z + N_{zs}u_s + Q_z w + M_{zz}\varphi_z + M_{zs}\varphi_s$ for the work of the edge stress resultants and couples N_{zz}, N_{zs}, Q_z, M_{zz}, M_{zs}.

It then follows from known results that an appropriate version of the principle of minimum complementary energy consists in the variational equation

$$\delta\left\{\int_{z_1}^{z_2}\int W\,\mathrm{d}s\,\mathrm{d}z - \int (N_{zz}\bar{u}_z + N_{zs}\bar{u}_s + Q_z\bar{w} + M_{zz}\bar{\varphi}_z + M_{zs}\bar{\varphi}_s)_{z_1}^{z_2}\,\mathrm{d}s\right\} = 0 \tag{19}$$

with the quantities $\bar{u}_z^{(1)}$ and $\bar{u}_z^{(2)}$, etc., being prescribed functions of s, and with the argument functions N_{zz}, etc., which enter into W being such as to satisfy the equilibrium differential eqns (1)–(6).

We note that the above statement for closed cross section shells contains the problem of the open cross section shell as a special case, by way of appropriate statements concerning the form of the complementary energy density function W and the distribution of stress resultants and couples, and that the results obtained in this way must come out the same as those directly derived for an open cross section shell with the stipulation of vanishing N_{ss}, N_{sz}, Q_s, M_{ss}, M_{sz} along the edges s = const.†

We also note that eqn (19) retains its validity if, at either edge, modifications of edge conditions, such as $u_z = \bar{u}_z$ into $N_{zz} = 0$, etc., are being assumed, except that in this event the stress states which are used in (19) must satisfy all the prescribed (homogeneous) stress resultant and couple boundary conditions as well.

FORM OF THE COMPLEMENTARY ENERGY FUNCTIONAL FOR RIGID BODY EDGE TRANSLATIONS AND ROTATIONS

We consider a cross section z = const, and consider separately in-plane and out-of-plane translations and rotations.

We write for out-of-plane translations and rotations

$$\bar{u}_z = U + \Phi_1 x_2 - \Phi_2 x_1,\qquad \bar{\varphi}_z = -\Phi_1 x_1' - \Phi_2 x_2', \tag{20}$$

where $x_1 = x_1(s)$, $x_2 = x_2(s)$, and U, Φ_1 and Φ_2 are given constants.

Insofar as in-plane translation and rotation are concerned, we first introduce *axes-parallel* translational components

$$\bar{u}_1 = U_1 - x_2\Phi,\qquad \bar{u}_2 = U_2 + x_1\Phi, \tag{21}$$

with U_1, U_2 and Φ being constants. Subsequent to this we write as expressions for tangential and normal translational components, and for the associated rotational component,

$$\bar{u}_s = \bar{u}_1 x_1' + \bar{u}_2 x_2',\qquad \bar{w} = \bar{u}_1 x_2' - \bar{u}_2 x_1',\qquad \bar{\varphi}_s = \Phi. \tag{22}$$

Introduction of (20) to (22) into the line integral portion of the complementary energy functional then gives

†And with the two conditions $M_{sz} = 0$ and $Q_s = 0$ being contracted into the one condition $Q_s + M_{sz,z} = 0$ for the case that W does not involve Q_s and Q_z.

$$\int (N_{zz}\bar{u}_z + \cdots + M_{zs}\bar{\varphi}_s)\,ds = \int \{N_{zz}[U + \Phi_1 x_2 - \Phi_2 x_1] - M_{zz}[\Phi_1 x_1' + \Phi_2 x_2'] + N_{zs}[(U_1 - x_2\Phi)x_1' + (U_2 + x_1\Phi)x_2'] + Q_z[(U_1 - x_2\Phi)x_2' - (U_2 + x_1\Phi)x_1'] + M_{zs}\Phi\}\,ds. \tag{23}$$

Equation (23) may be reduced further, with the help of the defining relations (7) to (12) and (14), so as to read

$$\int (N_{zz}\bar{u}_z + \cdots + M_{zs}\bar{\varphi}_s)\,ds = NU + M_1\Phi_1 + M_2\Phi_2 + F_1U_1 + F_2U_2 + T\Phi. \tag{24}$$

AXIALLY UNIFORM STATES OF STRESS

We consider, for the purpose of analyzing the problems of pure stretching, bending and twisting, the most general system of solutions of the equilibrium differential eqns (1)–(6) with the property that all resultants and couples are independent of the axial coordinate z.

We now have that that the system (1)–(6) decomposes into two separate systems, one of them being of the form

$$N_{ss}' + kQ_s = 0, \qquad Q_s' - kN_{ss} = 0, \qquad M_{ss}' = Q_s, \tag{25}$$

and the other being

$$N_{sz}' = 0, \qquad N_{zs} = N_{sz} + kM_{sz}, \qquad Q_z = M_{sz}'. \tag{26}$$

We note that these equations do not involve N_{zz}, M_{zz} and M_{zs}, and we further note that with these we have, on the basis of eqns (15) and (16), that

$$F_1 = 0, \qquad F_2 = 0, \tag{27}$$

in eqn (24), so that in fact the assumption of z-independent stress states implies a restriction to states of stretching, bending, and twisting alone, and the disappearance of the displacement components U_1 and U_2 from the line integral (24).

In order to fix the ideas in regard to the problem of pure stretching, bending, and twisting, we now consider a shell of length 2a, with rigid body translational and rotational displacement components

$$U = \pm a\epsilon, \qquad \Phi_i = \pm a\kappa_i, \qquad \Phi = \pm a\tau, \tag{28}$$

prescribed for the ends $z = \pm a$.

Introduction of (28) and (27) into (24), and observation of the fact that the axial uniformity of stress implies z-indepencence of the complementary energy density W in (19), then allows (19) to be reduced to

$$\delta\int \{W - N_{zz}\epsilon - (N_{zz}x_2 - M_{zz}x_1')\kappa_1 + (N_{zz}x_1 + M_{zz}x_2')\kappa_2 - (M_{zs} + M_{sz} + r_N N_{sz})\tau\}\,ds = 0, \tag{29}$$

with the variations of stress resultants and stress couples being independent except for restrictions imposed by the equilibrium eqns (25) and (26).

As regards these restrictions, we observe that the first two equations in (25) may be solved explicitly in terms of two constants N_1 and N_2, as follows

$$Q_s x_2' + N_{ss}x_1' = N_1, \qquad N_{ss}x_2' - Q_s x_1' = N_2. \tag{30}$$

Writing then

$$Q_s = N_1 x_2' - N_2 x_1', \qquad N_{ss} = N_1 x_1' + N_2 x_2', \tag{31}$$

the third equation in (25) gives, in terms of N_1 and N_2, and a third constant M,

$$M_{ss} = N_1 x_2 - N_2 x_1 + M. \tag{32}$$

We next solve the first equation in (26) in terms of a fourth constant S,

$$N_{sz} = S, \tag{33}$$

leaving the remaining two equations in (26) in the form

$$N_{zs} = S + kM_{sz}, \qquad Q_z = M_{sz}', \tag{34}$$

with the dependence of M_{sz} on s having to be determined through use of the variational eqn (29).

We will not now complete the indicated analysis in full generality, but rather restrict attention to the case of a shell with absent transverse shear deformability, and the twisting moment symmetry property $M_{sz} = M_{zs}$, and with the possibility of disregarding the difference between N_{sz} and N_{zs} in the constitutive equations of the problem. We then have that the complementary energy density function W is of the form

$$W = W(N_{zz}, N_{ss}, N_{sz}, M_{zz}, M_{ss}, M_{sz}), \tag{35}$$

and the variational eqn (29) becomes, in view of (31) to (33),

$$\int \left\{ \left(\frac{\partial W}{\partial N_{zz}} - \epsilon - \kappa_1 x_2 + \kappa_2 x_1 \right) \delta N_{zz} + \left(\frac{\partial W}{\partial M_{zz}} + \kappa_1 x_1' + \kappa_2 x_2' \right) \delta M_{zz} + \left(\frac{\partial W}{\partial M_{sz}} - 2\tau \right) \delta M_{sz} \right.$$
$$\left. + \left(\frac{\partial W}{\partial N_{sz}} - r_N \tau \right) \delta S + \frac{\partial W}{\partial N_{ss}} (x_1' \delta N_1 + x_2' \delta N_2) + \frac{\partial W}{\partial M_{ss}} (x_2 \delta N_1 - x_1 \delta N_2 + \delta M) \right\} \mathrm{d}s = 0. \tag{36}$$

Considering that δN_{zz}, δM_{zz} and δM_{sz} are arbitrary functions of s we obtain from (36) the three "local" Euler equations

$$\frac{\partial W}{\partial N_{zz}} = \epsilon + \kappa_1 x_2 - \kappa_2 x_1, \qquad \frac{\partial W}{\partial M_{zz}} = -\kappa_1 x_1' - \kappa_2 x_2', \qquad \frac{\partial W}{\partial M_{sz}} = 2\tau, \tag{37}$$

and in view of the fact that δS, δN_1, δN_2 and δM are four arbitrary constants we additionally obtain the four "global" Euler equations

$$\int \frac{\partial W}{\partial N_{sz}} \mathrm{d}s = \tau \int r_N \, \mathrm{d}s, \qquad \int \left(\frac{\partial W}{\partial N_{ss}} x_1' + \frac{\partial W}{\partial M_{ss}} x_2 \right) \mathrm{d}s = 0,$$
$$\int \left(\frac{\partial W}{\partial N_{ss}} x_2' - \frac{\partial W}{\partial M_{ss}} x_1 \right) \mathrm{d}s = 0, \qquad \int \frac{\partial W}{\partial M_{ss}} \mathrm{d}s = 0, \tag{38}$$

with these four equations being in the nature of kinematic single-valuedness conditions for the problem of the closed-cross section shell and, as can be shown by an appropriate limiting process, with a replacement of (38) by the relations

$$S = 0, \qquad N_1 = 0, \qquad N_2 = 0, \qquad M = 0, \tag{39}$$

for the open-cross section shell.

As regards the actual solution of the above general problem, the procedure will be to use (37) in order to express N_{zz}, M_{zz}, M_{sz} in terms of the eight constants ϵ, κ_1, κ_2, τ, S, N_1, N_2, M and to subsequently use the four defining eqns (7)–(9) and (17) for N, M_1, M_2 and T, in conjunction with the four global Euler equations in (38), for the simultaneous determination of the indicated eight constants in terms of N, M_1, M_2 and T. The results for four of these constants, in the form

$$\begin{aligned}\epsilon &= C_{\epsilon N}N + C_{\epsilon 1}M_1 + C_{\epsilon 2}M_2 + C_{\epsilon T}T,\\ \kappa_i &= C_{iN}N + C_{i1}M_1 + C_{i2}M_2 + C_{iT}T,\\ \tau &= C_{\tau N}N + C_{\tau 1}M_1 + C_{\tau 2}M_2 + C_{\tau T}T,\end{aligned} \tag{40}$$

then may be interpreted as a system of one-dimensional constitutive equations for the problem of combined stretching, bending, and twisting of cylindrical shell beams, subject to an assumption of negligibility of transverse shear deformation effects.

Application of the procedure as indicated, for the general case, comes out to be of some complexity. We have previously obtained the solution for the special case of pure torsion with complementary energy function $W = N_{sz}^2/2C + M_{sz}^2/2D$, with the result including an explicit torque-twist relation which contains as special cases the known classical results for thin-walled open and thin-walled closed cross sections[3]. We have also considered previously the special case of the problem of pure bending and stretching of cross-ply laminated shells[6]. We note here that for this case the function W comes out to be

$$W = \frac{N_{zz}^2 + N_{ss}^2 - 2\nu N_{zz}N_{ss}}{2(1-\lambda-\nu^2)C} + \frac{M_{zz}^2 + M_{ss}^2 - 2\nu M_{zz}M_{ss}}{2(1-\lambda-\nu^2)D} + \lambda\frac{M_{zz}N_{zz} - M_{ss}N_{ss}}{(1-\lambda-\nu^2)B}, \quad \lambda = \frac{B^2}{CD}, \tag{41}$$

and that with this function we may obtain an alternate derivation of the results in[6].

Constitutive equations for anisotropic thin-walled closed-cross-section shells

We consider now, as a further illustration of the general procedure, an approximate solution of the problem of combined stretching, bending, and twisting, with the approximation consisting in the assumption of the adequacy of a complementary energy density function W of the form

$$W = \frac{N_{zz}^2}{2C_E} + \frac{N_{sz}^2}{2C_G} + \frac{N_{zz}N_{sz}}{C_{EG}}. \tag{42}$$

With this expression for W eqns (37) reduce to the one equation

$$\frac{N_{zz}}{C_E} + \frac{N_{sz}}{C_{EG}} = \epsilon + \kappa_1 x_2 - \kappa_2 x_1, \tag{43}$$

and the consequences of eqns (38) are a relation

$$\int \left(\frac{N_{sz}}{C_G} + \frac{N_{zz}}{C_{EG}}\right) \mathrm{d}s = \tau \int r_N \,\mathrm{d}s, \tag{44}$$

with N_{sz} having the constant value S, in accordance with (33).

Additionally, we have from (7)–(9) and (17),

$$N = \int N_{zz}\,\mathrm{d}s, \qquad M_1 = \int N_{zz}x_2\,\mathrm{d}s, \qquad M_2 = -\int N_{zz}x_1\,\mathrm{d}s, \tag{45}$$

$$T = \int N_{sz}r_N\,\mathrm{d}s. \tag{46}$$

We now use (43) and (44) in order to obtain as expression for the shear-stress resultant S,

$$S\int\left(1-\frac{C_E C_G}{C_{EG}}\right)\frac{\mathrm{d}s}{C_G}=\tau\int r_N\,\mathrm{d}s-\epsilon\int\frac{C_E\,\mathrm{d}s}{C_{EG}}-\kappa_1\int x_2\frac{C_E\,\mathrm{d}s}{C_{EG}}+\kappa_2\int x_1\frac{C_E\,\mathrm{d}s}{C_{EG}}, \tag{47}$$

with the torque T then given in terms of τ, ϵ, κ_1 and κ_2, in accordance with (46), in the form

$$T=S\int r_N\,\mathrm{d}s, \tag{48}$$

and with the axial force N and the bending moments M_1 and M_2 given as

$$\begin{aligned}
N&=-S\int\frac{C_E\,\mathrm{d}s}{C_{EG}}+\epsilon\int C_E\,\mathrm{d}s+\kappa_1\int C_E x_2\,\mathrm{d}s-\kappa_2\int C_E x_1\,\mathrm{d}s,\\
M_1&=-S\int x_2\frac{C_E\,\mathrm{d}s}{C_{EG}}+\epsilon\int C_E x_2\,\mathrm{d}s+\kappa_1\int C_E x_2^2\,\mathrm{d}s-\kappa_2\int C_E x_2 x_1\,\mathrm{d}s,\\
M_2&=+S\int x_1\frac{C_E\,\mathrm{d}s}{C_{EG}}-\epsilon\int C_E x_1\,\mathrm{d}s-\kappa_1\int C_E x_1 x_2\,\mathrm{d}s+\kappa_2\int C_E x_2^2\,\mathrm{d}s.
\end{aligned} \tag{49}$$

It is evident that relations of the form (40) will result upon inversion of (48) and (49), with S as in (47), and that a considerable simplification of the results occurs upon introducing the stipulations $\int(x_1, x_2, x_1x_2)C_E\,\mathrm{d}s=0$ and/or $\int(x_1, x_2)(C_E/C_{EG})\,\mathrm{d}s=0$.

THE PROBLEM OF FLEXURE AND THE CENTERS OF SHEAR AND OF TWIST

We now consider a shell of length a with the end $z=0$ held fixed so that $\bar{u}_z=\bar{u}_s=\bar{w}=\bar{\varphi}_z=\bar{\varphi}_s=0$ for $z=0$. Insofar as the end $z=a$ is concerned we prescribe $N_{zz}=0$ and $M_{zz}=0$ in place of conditions on u_z and φ_z, with the remaining conditions being those of in-plane rigid body translation and rotation, in accordance with eqns (21) and (22).

The appropriate version of the principle of minimum complementary energy follows then from eqns (19) and (24) in the form

$$\delta\left\{\int_0^a\int W\,\mathrm{d}s\,\mathrm{d}z-F_1U_1-F_2U_2-T\Phi\right\}=0, \tag{50}$$

with F_1, F_2 and T as before given by eqns (10)–(12).

Before attempting to evaluate the variational eqn (50) we recall our previously proposed definitions for shear and twist center locations. These are based on the evident fact that, whatever the solution of (50) in conjunction with the equilibrium eqns (1)–(6) and the boundary conditions $N_{zz}(a,s)=M_{zz}(a,s)=0$ turns out to be, it will imply expressions for forces F_i and torque T, in terms of stiffness coefficients K, of the form

$$F_i=K_{ij}U_j+K_{i\Phi}\Phi,\qquad T=K_{Tj}U_j+K_{T\Phi}\Phi, \tag{51}$$

and therewith also inverted relations, in terms of flexibility coefficients C, of the form

$$U_i=C_{ij}F_j+C_{iT}T,\qquad \Phi=C_{\Phi j}F_j+C_{\Phi T}T. \tag{52}$$

With this, and with eqns (21) for the displacements $\bar{u}_i$ in the direction of the axes x_i of all the points of the end cross sections, we now define the coordinates x_{is} of a *center of shear* as the coordinates of that point through which the lines of action of the forces F_i must pass in order that there be no rotation of the end cross section, with the torque T being just the moment of the forces F_i about the origin of the x_i-coordinate system.

Setting then, in accordance with this definition, simultaneously

$$\Phi=0,\qquad T=F_2x_{1s}-F_1x_{2s} \tag{53}$$

in the third relation in (52), we will have

$$(C_{\Phi 1} - x_{2s}C_{\Phi T})F_1 + (C_{\Phi 2} + x_{1s}C_{\Phi T})F_2 = 0. \tag{54}$$

Since (54) must hold independent of the value of the ratio F_1/F_2 we obtain as expressions for the coordinates of the center of shear

$$x_{1s} = -\frac{C_{\Phi 2}}{C_{\Phi T}}, \qquad x_{2s} = \frac{C_{\Phi 1}}{C_{\Phi T}}. \tag{55}$$

Having eqns (52), it is equally well possible to obtain the location of a *center of twist* in terms of the flexibility coefficients C. We define the coordinates x_{iT} of the center of twist as the coordinates of that point in the cross section which remains fixed, in the event that the end section load consists solely of a torque T, with both components of force F_i having zero values.

Setting then, in accordance with this definition

$$F_1 = F_2 = 0, \qquad U_1 - x_{2T}\Phi = U_2 + x_{1T}\Phi = 0, \tag{56}$$

we have, on the basis of eqns (52),

$$x_{iT} = -\frac{C_{2T}}{C_{\Phi T}}, \qquad x_{2T} = \frac{C_{1T}}{C_{\Phi T}}. \tag{57}$$

We note that while the coordinates of the center of shear are given in terms of the elements of the third *row* of the flexibility coefficient matrix, we have that the coordinates of the center of twist are given in terms of the elements of the third *column* of this matrix, in such a way that center of shear and center of twist coincide for the case of a *symmetric* flexibility coefficient matrix.

We now turn to the problem of selecting expressions for stress resultants and couples for use in conjunction with the variational equation (50). Evidently, for an exact solution of the problem it would be necessary to use the most general solution of the differential-equation system (1)–(6). While it is known how to write this most general solution in terms of stress functions, introduction of these stress function expressions into the variational equation would, in the end, leave us with a boundary value problem for a system of partial differential equations, of a kind which could equally well have been established without use of variational considerations. The *advantages* of the variational procedure are that with it we may utilize suitably *selected* solutions of eqns (1)–(6), in the expectation that thereby a rational approximate solution of the problem of flexure, as formulated in the foregoing, may be obtained.

Our primary selective endeavor in what follows is to limit our choice of expressions for stress resultants and couples by the stipulation that those resultants and couples which enter into the expressions for forces F_i and torque T are independent of the axial coordinate z, just as the values of F_i and T themselves.

Setting then

$$N_{zs} = N_{zs}(s), \qquad Q_z = Q_z(s), \qquad M_{zs} = M_{zs}(s), \tag{58}$$

we note first that thereby eqns (1), (3) and (5) are reduced to the same form (25) for N_{ss}, Q_s, and M_{ss} as for the case of axially uniform states of stress. This means that we must again take N_{ss}, Q_s, and M_{ss} in the form (31) and (32), except that now the parameters N_1, N_2, and M may be functions of z which remain to be determined.

An inspection of the remaining eqns (2), (4) and (6) suggests further that we complement eqns (49) by assumptions of the form

$$N_{sz} = S(s), \qquad M_{sz} = R(s), \tag{59}$$

where introduction of the symbols S and R will be convenient for what follows.

Equations (59) in conjunction with the conditions that $N_{zz}(a, s) = M_{zz}(a, s) = 0$ in turn imply that

$$N_{zz} = (a - z)S', \qquad N_{zs} = S + kR, \qquad M_{zz} = (a - z)(R' - Q_z), \tag{60}$$

and it now only remains to use the variational eqn (50), with or without additional constraints imposed on the form of the functions S, R, Q_z, M_{zs} in (58)–(60) and on the form of the functions N_1, N_2, and M in (31) and (32), for the sake of deriving Euler differential equations and boundary conditions for these functions. In what follows we shall be concerned with two aspects of this problem, which seem to us to be of particular interest.

EFFECT OF ANISOTROPY ON SHEAR CENTER LOCATION: AN EXPLICIT FORMULA

In order to obtain an indication of the effect of anisotropy in the solution of the problem of flexure, we consider the problem of determing values of the flexibility coefficients in eqns (52) for a shell with complementary energy density function

$$W = \frac{N_{zz}^2}{2C_E} + \frac{N_{sz}^2}{2C_G} + \frac{N_{zz}N_{sz}}{C_{EG}} + \frac{M_{sz}^2}{2D}, \tag{61}$$

with the choice of this function being associated by us with an assumption of negligible contributions of N_{ss}, M_{ss} and M_{zz}, with an assumption of negligible transverse shear deformability of the elements of the shell, and with the assumption that $M_{zs} = M_{sz}$.

Neglecting M_{zz} means additionally that we now have for the forces F_i and toruqe T in eqn (50) the integral representations

$$F_i = \oint N_{sz}x_i' \, ds, \qquad T = \oint (2M_{sz} + r_N N_{sz}) \, ds. \tag{62}$$

We initiate our calculations by a St. Venant-type assumption for the distribution of axial normal stress, in the form

$$N_{zz} = (z - a)[F_1 n_1(s) + F_2 n_2(s)], \tag{63}$$

with the functions $n_i(s)$ at this stage being subject to no other requirement than to lead to results which are consistent with eqns (7)–(9), in which now $M_1 = (z - a)F_2$ and $M_2 = -(z - a)F_1$. These requirements are satisfied provided

$$\oint n_i \, ds = 0, \qquad \oint n_i x_j \, ds = \delta_{ij}. \tag{64}$$

We next use (63) in order to determine N_{sz}, consistent with eqns (59) and (60), in the form

$$N_{sz} = S_0 - F_1 \int_0^s n_1 \, ds - F_2 \int_0^s n_2 \, ds, \tag{65}$$

with S_0 being a constant of integration.

It will turn out to be of advantage to express S_0 in terms of a contribution $T_N = \oint N_{sz} r_N \, ds$ to the total torque T, and to write

$$N_{sz} = T_N m_0 + F_1 m_1(s) + F_2 m_2(s), \tag{66}$$

where

$$m_0 = \frac{1}{\oint r_N \, ds}, \qquad m_i = \frac{\oint \left(\int_0^s n_i \, ds \right) r_N \, ds}{\oint r_N \, ds} - \int_0^s n_i \, ds, \quad i = 1, 2. \tag{67}$$

We introduce eqns (63) and (67) into eqn (61), and the result into the variational eqn (50). In this way eqn (50) assumes the form

$$\delta\left\{\oint\left[\frac{a^3}{3}\frac{(F_1n_1+F_2n_2)^2}{2C_E}+a\frac{(T_Nm_0+F_1m_1+F_2m_2)^2}{2C_G}\right.\right.$$

$$-\frac{a^2}{2}\frac{(F_1n_1+F_2n_2)(T_Nm_0+F_1m_1+F_2m_2)}{C_{EG}}$$

$$\left.\left.+a\frac{M_{sz}^2}{2D}\right]\mathrm{d}s-F_1U_1-F_2U_2-\left(T_N+\oint 2M_{sz}\,\mathrm{d}s\right)\Phi\right\}=0, \tag{68}$$

with the independent variations in this equation being the three constants δF_i and δT_N, and the one function δM_{sz}.

The Euler equation coming from δM_{sz} is

$$aM_{sz}=2D\Phi. \tag{69}$$

The Euler equations coming from δF_i and δT_N are

$$\oint\left[\frac{a^3}{3}\frac{F_1n_1+F_2n_2}{C_E}n_i+a\frac{T_Nm_0+F_1m_1+F_2m_2}{C_G}m_i\right.$$

$$\left.-\frac{a^2}{2}\frac{(F_1n_1+F_2n_2)m_i+(T_Nm_0+F_1m_1+F_2m_2)n_i}{C_{GE}}\right]\mathrm{d}s=U_i, \tag{70}$$

$$\oint\left[a\frac{T_Nm_0+F_1m_1+F_2m_2}{C_G}m_0-\frac{a^2}{2}\frac{F_1n_1+F_2n_2}{C_{EG}}m_0\right]\mathrm{d}s=\Phi. \tag{71}$$

Having obtained eqns (70) and (71), we very nearly have arrived at the desired result consisting of expressions for Φ and the U_i in terms of T and the F_i, as in eqn (52). To complete the analysis we write the second relation in (62) in the form $T=T_N+T_M$ where $T_M=\oint 2M_{sz}\,\mathrm{d}s$. From this we obtain, with the help of (69)

$$T_N=T-T_M=T-4a^{-1}\Phi\oint D\,\mathrm{d}s. \tag{72}$$

Introduction of (72) into (71) gives a result which is equivalent to the third relation in (52), in the form

$$F_1\oint\left(\frac{m_1}{C_G}-\frac{an_1}{2C_{EG}}\right)\frac{\mathrm{d}s}{m_0}+F_2\oint\left(\frac{m_2}{C_G}-\frac{an_2}{2C_{EG}}\right)\frac{\mathrm{d}s}{m_0}+T\oint\frac{\mathrm{d}s}{C_G}=\frac{\Phi}{a}\left(\frac{1}{m_0^2}+4\oint D\,\mathrm{d}s\oint\frac{\mathrm{d}s}{C_G}\right). \tag{73}$$

We omit listing explicit expressions for the flexibility coefficients $C_{\Phi j}$ and $C_{\Phi T}$ which are implied by (73), as well as for the flexibility coefficients C_{ij} and C_{iT} which are implied by (70) in conjunction with (73), and the first relation in (43). We will however state explicitly expressions for the coefficient ratios which determine the location of the center of shear, in accordance with eqns (55). Evidently these expressions are

$$x_{1s}=-\frac{\oint\left(\frac{m_2}{C_G}-\frac{an_2}{2C_{EG}}\right)\mathrm{d}s}{\oint\frac{m_0}{C_G}\mathrm{d}s},\qquad x_{2s}=\frac{\oint\left(\frac{m_1}{C_G}-\frac{an_1}{2C_{EG}}\right)\mathrm{d}s}{\oint\frac{m_0}{C_G}\mathrm{d}s}. \tag{74}$$

With m_0 and n_i given in accordance with (63), (64) and (67), we may write alternately,

$$x_{1s}=-\oint\left(\int_0^s n_2\,\mathrm{d}s\right)r_N\,\mathrm{d}s+\frac{\oint r_N\,\mathrm{d}s\oint\left(\int_0^s n_2\,\mathrm{d}s\right)(\mathrm{d}s/C_G)}{\oint(\mathrm{d}s/C_G)}+\frac{a\oint r_N\,\mathrm{d}s}{\oint(\mathrm{d}s/C_G)}\oint\frac{n_2}{C_{EG}}\,\mathrm{d}s, \quad (75)$$

with a corresponding equation for x_{2s}.

Remarkably, while eqn (75) is valid for the case of a closed-cross section we may from it deduce directly the elementary result for the case of the open-cross section, as follows. Let l_0 be the circumferential length of the open-cross section and $l_c > l_0$ the corresponding length of a fictitious closed-cross section such that $C_G = 0$ for $l_0 < s \le l_c$. Considering the fact that in the interval (l_0, l_c) we will also have $n_2 = 0$ and $\int_{l_0}^{s} n_2\,\mathrm{d}s = 0$, with this latter result being a consequence of eqns (64), we may conclude that for the open-cross section the second and third terms in (75) take on the values zero and so, for an open-cross section

$$x_{1s}=-\int_0^{l_0}\left(\int_0^s n_2\,\mathrm{d}s\right)r_N\,\mathrm{d}s. \quad (75')$$

again with a corresponding equation for x_{2s}.

Shear center coordinates for St. Venant normal stress distribution

We now assume that the coordinate axes are principal centroidal axes in the sense that

$$\oint C_E x_i\,\mathrm{d}s = 0, \qquad \oint C_E x_1 x_2\,\mathrm{d}s = 0, \quad (76)$$

and we choose as expressions for the functions n_i in eqn (63) for the axial normal stress resultant distribution N_{zz},

$$I_i n_i = C_E x_i, \qquad I_i = \oint C_E x_i^2\,\mathrm{d}s, \quad (77)$$

with (77), in conjunction with (76), being consistent with (64), as it must be.

Introduction of n_2 from (77) into eqn (66) gives further

$$x_{1s}=\frac{1}{I_2}\left\{\frac{\oint r_N\,\mathrm{d}s}{\oint(\mathrm{d}s/C_G)}\oint\left(\int_0^s C_E x_2\,\mathrm{d}s\right)\frac{\mathrm{d}s}{C_G}-\oint\left(\int_0^s C_E x_2\,\mathrm{d}s\right)r_N\,\mathrm{d}s+\frac{\oint r_N\,\mathrm{d}s}{\oint \mathrm{d}s/C_G}\oint\frac{aC_E x_2}{C_{EG}}\,\mathrm{d}s\right\}, \quad (78)$$

again with a corresponding expression for x_{2s}.

We note, for the sake of a comparison with earlier results in the literature [7], the possibility of integrating by parts, in the form

$$\oint\left(\int_0^s C_E x_2\right)\frac{\mathrm{d}s}{C_G}=0-\oint\left(\int_0^s\frac{\mathrm{d}s}{C_G}\right)C_E x_2\,\mathrm{d}s,$$
$$\oint\left(\int_0^s C_E x_2\right)r_N\,\mathrm{d}s=0-\oint\left(\int_0^s r_N\,\mathrm{d}s\right)C_E x_2\,\mathrm{d}s. \quad (79)$$

With this, and upon setting $1/C_{EG} = 0$, $C_G = Gt$ and $C_E = Et$, with G and E independent of s, eqn (78) and its open-cross section specialization (78′) reduce to formulas previously given in [7].

We also note the possibility of using (75) for the purpose of deducing values of x_{1s}, consistent with St. Venant's assumption $n_2 = C_E(A_{02} + A_{12}x_1 + A_{22}x_2)$, for other than principal centroidal axes. A necessary step for this is to determine the three coefficients A_{i2} through use of three relations $\oint(1, x_1, x_2)n_2\,ds = (0, 0, 1)$ which are part of eqn (64).

EFFECT OF SHEAR LAG ON SHEAR CENTER LOCATION

We define shear lag as the influence which transverse shear deformation in beams has on the distribution curves for stress, in comparison with the corresponding results when this effect is being neglected. Earlier studies of the problem of shear lag have shown its importance in particular with reference to "wide-flanged" beams [2]. However, no consideration is known to us concerning the influence of this effect on shear and twist center location.

In what follows we show the nature of the analysis which is involved, for simplicity's sake limited to the open-cross section case. Additionally, we obtain explicit results for the modification due to shear lag of the well-known elementary formula for the location of the center of shear of a beam with uniform thin-walled circular ring section cross section, as given first by Griffith and Taylor [1].

We once again depart from the variational eqn (41), this time for a cylindrical shell with complementary energy density

$$W = \frac{N_{zz}^2}{2C_E} + \frac{N_{sz}^2}{2C_G} + \frac{M_{sz}^2}{2D}. \tag{80}$$

We make no assumption concerning the distribution of N_{zz} over the cross section, as in (63), but rather use eqn (59), and the first relation in (60), with the variational eqn (50) now becoming

$$\delta\int_{s_1}^{s_2}\left\{\frac{a^3}{3}\frac{(S')^2}{2C_E} + a\frac{(S)^2}{2C_G} + a\frac{(R)^2}{2D} - (U_1x_1' + U_2x_2' + \Phi r_N)S - 2\Phi R\right\}ds = 0, \tag{81}$$

and with the function S subject to the constraint boundary conditions $S(s_1) = S(s_2) = 0$.

Evidently, the Euler equations associated with (81) are the two differential equations, of second and zeroth order, respectively,

$$-\frac{a^3}{3}\left(\frac{S'}{C_E}\right)' + a\frac{S}{C_G} = U_1x_1' + U_2x_2' + \Phi r_N, \tag{82}$$

$$aR = 4\Phi D, \tag{83}$$

with F_i and T in eqn (51) now being of the form

$$F_i = \int_{s_1}^{s_2} Sx_i'\,ds, \qquad T = \int_{s_1}^{s_2}(2R + r_NS)\,ds. \tag{84}$$

Solution for uniform circular ring-sector cross section

We write, in accordance with Fig. 2,

$$x_1 = b\cos s/b, \qquad x_2 = b\sin s/b, \tag{85}$$

and therewith

$$x_1' = -\sin s/b, \qquad x_2' = \cos s/b, \qquad r_N = b, \tag{86}$$

with s confined to the interval $-\beta b \le s \le \beta b$.

Assuming C_E and C_G to be independent of s we have as solution of the differential equation (82) subject to the boundary conditions $S(\pm\beta b) = 0$,

$$S = \frac{C_G}{a}\left\{b\Phi\left(1 - \frac{\cosh\lambda s/b}{\cosh\lambda\beta}\right) + \frac{\lambda^2U_2}{1+\lambda^2}\left(\cos\frac{s}{b} - \cos\beta\frac{\cosh\lambda s/b}{\cosh\lambda\beta}\right)\right.$$
$$\left. - \frac{\lambda^2U_1}{1+\lambda^2}\left(\sin\frac{s}{b} - \sin\beta\frac{\sinh\lambda s/b}{\sinh\lambda\beta}\right)\right\}, \tag{87}$$

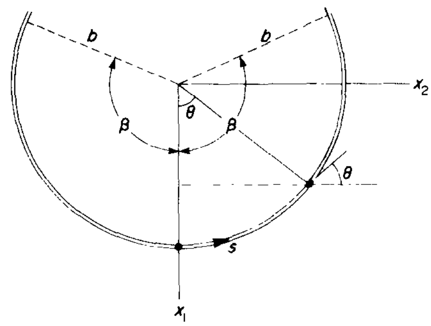

Fig. 2.

where

$$\lambda = \sqrt{(3C_E/C_G)}\, b/a. \tag{88}$$

Introduction of (87) and (86) into eqn (84) gives as expressions for forces and torque

$$F_1 = -\frac{3C_E U_1}{1+\lambda^2}\frac{b^3}{a^3}\left\{\beta - \frac{\sin 2\beta}{2} + \frac{\sin\beta}{1+\lambda^2}\left(\cos\beta - \frac{\lambda\sin\beta}{\tanh\lambda\beta}\right)\right\}, \tag{89}$$

$$F_2 = \frac{3C_E U_2}{1+\lambda^2}\frac{b^3}{a^3}\left\{\beta - \frac{\sin 2\beta}{2} + \frac{2\lambda^2\cos\beta}{1+\lambda^2}\left(\sin\beta - \cos\beta\frac{\tanh\lambda\beta}{\lambda}\right)\right\}$$
$$+\frac{6C_E b\Phi}{1+\lambda^2}\frac{b^3}{a^3}\left\{\sin\beta - \cos\beta\frac{\tanh\lambda\beta}{\lambda}\right\}, \tag{90}$$

$$T = \frac{6C_E bU_2}{1+\lambda^2}\frac{b^3}{a^3}\left\{\sin\beta - \cos\beta\frac{\tanh\lambda\beta}{\lambda}\right\} + \frac{\Phi}{a}\left\{2C_G b^3\beta\left(1 - \frac{\tanh\lambda\beta}{\lambda\beta}\right) + 4Db\beta\right\}. \tag{91}$$

We note that the symmetry of the cross section about the x_1-axis leads, as expected, to the result that F_1 depends on U_1 alone, while F_2 and T depend on both U_2 and Φ, with the flexibility coefficients in (90) and (91) satisfying the expected symmetry relation $K_{2\Phi} = K_{T2}$.

In determining an expression for the shear center coordinate x_{1s} we find that the assumed symmetry makes it just as easy to express x_{1s} in terms of stiffness coefficients, in place of using the general flexibility coefficient formula (55). Setting $\Phi = 0$ and $T = F_2 x_{1s}$ we now have from (90) and (91)

$$\frac{x_{1s}}{2b} = \frac{\sin\beta - \cos\beta\dfrac{\tanh\lambda\beta}{\lambda}}{\beta - \sin\beta\cos\beta + \dfrac{2\lambda^2\cos\beta}{1+\lambda^2}\left(\sin\beta - \cos\beta\dfrac{\tanh\lambda\beta}{\lambda}\right)}. \tag{92}$$

Considering the defining relation (88) for λ, and the fact that the nature of our approximation is such as to limit the validity of our results to the case of relatively small values of λ, we may replace eqn (92) effectively by

$$\frac{x_{1s}}{2b} = \frac{\sin\beta - \beta\cos\beta + \frac{1}{3}\lambda^2\beta^3\cos\beta}{\beta - \sin\beta\cos\beta + 2\lambda^2\cos\beta(\sin\beta - \beta\cos\beta)}, \tag{93}$$

with the well-known elementary theory formula resulting from (93) upon setting $\lambda = 0$ in this equation.

We note in particular the special-case results

$$\beta = \pi, \quad \frac{x_{1s}}{2b} = \frac{1 - \pi^2\lambda^2/3}{1 - 2\lambda^2} \approx 1 - \left(\frac{\pi^2}{3} - 2\right)\lambda^2 \approx 1 - 4\frac{C_E}{C_G}\frac{b^2}{a^2},$$
$$\beta = \frac{\pi}{2}, \quad \frac{x_{1s}}{2b} = \frac{2}{\pi}\,(\text{independent of } \lambda). \tag{94}$$

The possible magnitude of the effect of shear lag on the location of the center of shear may be gaged from a numerical example, with $b/a = 0.1$, and $C_E/C_G = E/G = 2(1+\nu) = 8/3$, giving a ratio $x_{1s}/2b \approx 1 - 0.1 = 0.9$ in place of the conventional value 1.0.

We will refrain from discussing further implications of the force deflection relations (89)–(91), except for stating the simple formula

$$\beta = \pi, \qquad F_1 = -\frac{3C_E U_1 \pi}{1+\lambda^2}\frac{b^3}{a^3} \approx -3C_E U_1 \frac{\pi b^3}{a^3}\left(1 - 3\frac{C_E}{C_G}\frac{b^2}{a^2}\right), \tag{95}$$

in which the additional deflection due to transverse shear is represented by the second term in the parenthesis on the right.

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